Chapter 1

Introduction

The last thing one discovers in
writing a book is what to put first.
Blaise Pascal

In this introductory chapter we provide an intuitive background to the material that we present more formally in later chapters. Terms that appear here in bold-face type are to be thought of as descriptions rather than as definitions. Having met them here in an informal setting, you should find them more familiar when you meet them later. So read this chapter quickly, and then forget all about it!

1 What is a graph?

We begin by considering Figs. 1.1 and 1.2, which depict part of a road map and part of an electrical network.

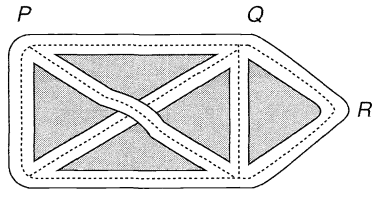

Fig. 1.1

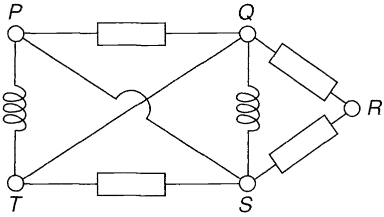

Fig. 1.2

Either of these situations can be represented diagrammatically by means of points and lines, as in Fig. 1.3. The points P, Q, R, S and T are called **vertices**, the lines are called **edges**, and the whole diagram is called a **graph**. Note that the intersection of the lines PS and QT is not a vertex, since it does not correspond to a cross-roads or to the meeting of two wires. The **degree** of a vertex is the number of edges with that vertex as an end-point; it corresponds in Fig. 1.1 to the number of roads at an intersection. For example, the degree of the vertex Q is 4.

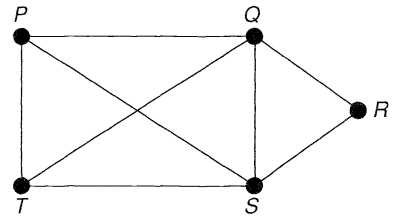

Fig. 1.3

The graph in Fig. 1.3 can also represent other situations. For example, if P, Q, R, S and T represent football teams, then the existence of an edge might correspond to the playing of a game between the teams at its end-points. Thus, in Fig. 1.3, team P has played against teams Q, S and T, but not against team R. In this representation, the degree of a vertex is the number of games played by the corresponding team.

Another way of depicting these situations is by the graph in Fig. 1.4. Here we have removed the 'crossing' of the lines PS and QT by drawing the line PS outside the rectangle $PQST$. The resulting graph still tells us whether there is a direct road from one intersection to another, how the electrical network is wired up, and which football teams have played which. The only information we have lost concerns 'metrical' properties, such as the length of a road and the straightness of a wire.

Thus, a graph is a representation of a set of points and of how they are joined up, and any metrical properties are irrelevant. From this point of view, any graphs that represent the same situation, such as those of Figs. 1.3 and 1.4, are regarded as the same graph.

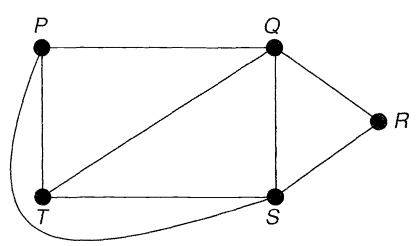

Fig. 1.4

More generally, two graphs are the same if two vertices are joined by an edge in one graph if and only if the corresponding vertices are joined by an edge in the other. Another graph that is the same as the graphs in Figs. 1.3 and 1.4 is shown in Fig. 1.5. Here all idea of space and distance has gone, although we can still tell at a glance which points are joined by a road or a wire.

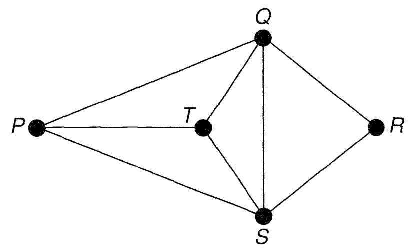

Fig. 1.5

In the above graph there is at most one edge joining each pair of vertices. Suppose now, that in Fig. 1.5 the roads joining Q and S, and S and T, have too much traffic to carry. Then the situation is eased by building extra roads joining these points, and the resulting diagram looks like Fig. 1.6. The edges joining Q and S, or S and T, are called **multiple edges**. If, in addition, we need a car park at P, then we indicate this by drawing an edge from P to itself, called a **loop** (see Fig. 1.7). In this book, a graph may contain loops and multiple edges. Graphs with no loops or multiple edges, such as the graph in Fig. 1.5, are called **simple graphs**.

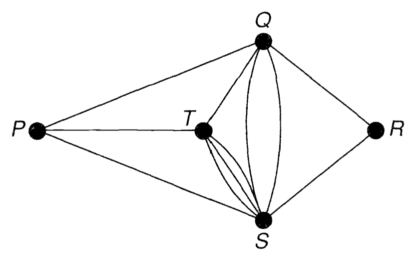

Fig. 1.6

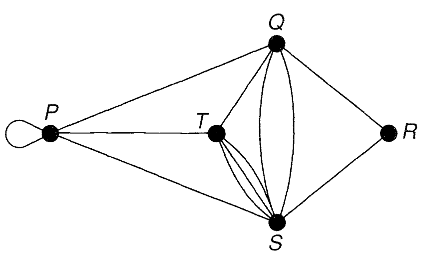

Fig. 1.7

The study of **directed graphs** (or **digraphs,** as we abbreviate them) arises from making the roads into one-way streets. An example of a digraph is given in Fig. 1.8, the directions of the one-way streets being indicated by arrows. (In this example, there would be chaos at T, but that does not stop us from studying such situations!) We discuss digraphs in Chapter 7.

Much of graph theory involves 'walks' of various kinds. A **walk** is a 'way of getting from one vertex to another', and consists of a sequence of edges, one following after another. For example, in Fig 1.5 $P \to Q \to R$ is a walk of length 2, and $P \to S \to Q \to T \to S \to R$ is a walk of length 5. A walk in which no vertex appears more than once is

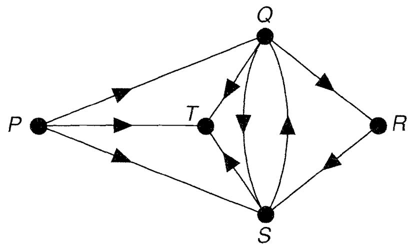

Fig. 1.8

called a **path**; for example, $P \to T \to S \to R$ is a path. A walk of the form $Q \to S \to T \to Q$ is called a **cycle**.

Much of Chapter 3 is devoted to walks with some special property. In particular, we discuss graphs containing walks that include every edge or every vertex exactly once, ending at the initial vertex; such graphs are called **Eulerian** and **Hamiltonian** graphs, respectively. For example, the graph in Figs 1.3–1.5 is Hamiltonian; a suitable walk is $P \to Q \to R \to S \to T \to P$. It is not Eulerian, since any walk that includes each edge exactly once (such as $P \to Q \to R \to S \to T \to P \to S \to Q \to T$) must end at a vertex different from the initial one.

Some graphs are in two or more parts. For example, consider the graph whose vertices are the stations of the London Underground and the New York Subway, and whose edges are the lines joining them. It is impossible to travel from Trafalgar Square to Grand Central Station using only edges of this graph, but if we confine our attention to the London Underground only, then we can travel from any station to any other. A graph that is in one piece, so that any two vertices are connected by a path, is a **connected graph**; a graph in more than one piece is a **disconnected graph** (see Fig. 1.9). We discuss connectedness in Chapter 3.

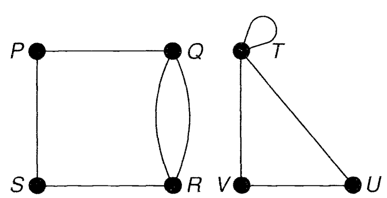

Fig. 1.9

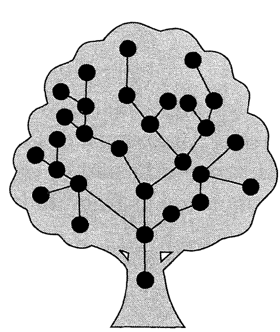

Fig. 1.10

We are sometimes interested in connected graphs with only one path between each pair of vertices. Such graphs are called **trees**, generalizing the idea of a family tree, and are considered in Chapter 4. As we shall see, a tree can be defined as a connected graph containing no cycles (see Fig. 1.10).

Earlier we noted that Fig. 1.3 can be redrawn as in Figs 1.4 and 1.5 so as to avoid crossings of edges. A graph that can be redrawn without crossings in this way is called a **planar graph**. In Chapter 5 we give several criteria for planarity. Some of these involve the properties of 'subgraphs' of the graph in question; others involve the fundamental notion of duality.

Planar graphs also play an important role in colouring problems. In our 'road-map' graph, let us suppose that Shell, Esso, BP, and Gulf wish to erect five garages between them, and that for economic reasons no company wishes to erect two garages at neighbouring corners. Then Shell can build at P, Esso can build at Q, BP can build at S, and Gulf can build at T, leaving either Shell or Gulf to build at R (see Fig. 1.11). However, if Gulf backs out of the agreement, then the other three companies cannot erect the garages in the specified manner.

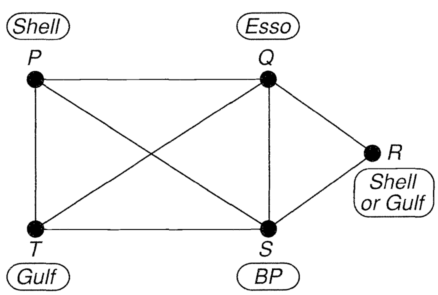

Fig. 1.11

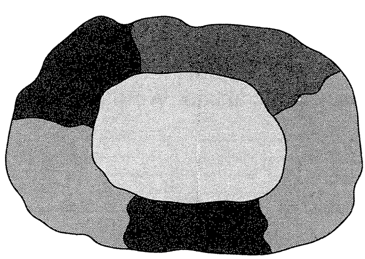

Fig. 1.12

We discuss such problems in Chapter 6, where we try to colour the vertices of a simple graph with a given number of colours so that each edge of the graph joins vertices of different colours. If the graph is planar, then we can always colour its vertices in this way with only four colours – this is the celebrated **four-colour theorem**. Another version of this theorem is that we can always colour the countries of any map with four colours so that no two neighbouring countries share the same colour (see Fig. 1.12).

In Chapter 8 we investigate the celebrated **marriage problem**, which asks under what conditions a collection of girls, each of whom knows several boys, can be married

so that each girl marries a boy she knows. This problem can be expressed in the language of 'transversal theory', and is related to problems of finding disjoint paths connecting two given vertices in a graph or digraph.

Chapter 8 concludes with a discussion of network flows and transportation problems. Suppose that we have a transportation network such as in Fig. 1.13, in which P is a factory, R is a market, and the edges of the graph are channels through which goods can be sent. Each channel has a capacity, indicated by a number next to the edge, representing the maximum amount that can pass through that channel. The problem is to determine how much can be sent from the factory to the market.

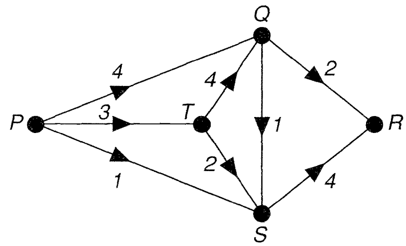

Fig. 1.13

We conclude with a chapter on matroids. This ties together the material of the previous chapters, while satisfying the maxim 'be wise – generalize!' Matroid theory, the study of sets with 'independence structures' defined on them, generalizes both linear independence in vector spaces and some results on graphs and transversals from earlier in the book. However, matroid theory is far from being 'generalization for generalization's sake'. On the contrary, it gives us deeper insight into several graph problems, as well as providing simple proofs of results on transversals that are awkward to prove by more traditional methods. Matroids have played an important role in the development of combinatorial ideas in recent years.

We hope that this introductory chapter has been useful in setting the scene and describing some of the treats that lie ahead. We now embark upon a formal treatment of the subject.

Exercises 1

1.1[s] Write down the number of vertices, the number of edges, and the degree of each vertex, in:

(i) the graph in Fig. 1.3;
(ii) the tree in Fig. 1.14.

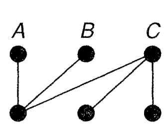

Fig. 1.14

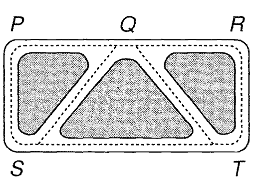

Fig. 1.15

1.2 Draw the graph representing the road system in Fig. 1.15, and write down the number of vertices, the number of edges and the degree of each vertex.

1.3[s] Figure 1.16 represents the chemical molecules of methane ($\mathbf{CH_4}$) and propane ($\mathbf{C_3H_8}$).

(i) Regarding these diagrams as graphs, what can you say about the vertices representing carbon atoms (**C**) and hydrogen atoms (**H**)?

(ii) There are two different chemical molecules with formula $\mathbf{C_4H_{10}}$. Draw the graphs corresponding to these molecules.

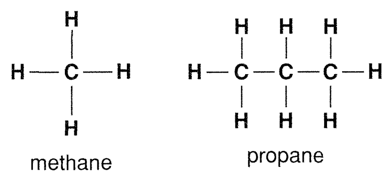

Fig. 1.16

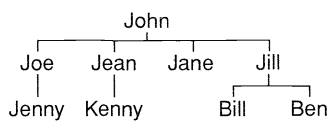

Fig. 1.17

1.4 Draw a graph corresponding to the family tree in Fig. 1.17.

1.5* Draw a graph with vertices $A, \ldots, M$ that shows the various routes one can take when tracing the Hampton Court maze in Fig. 1.18.

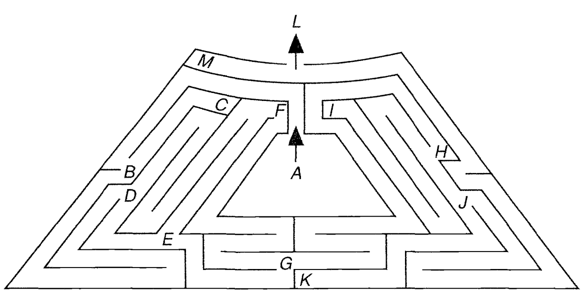

Fig. 1.18

1.6[s] John likes Joan, Jean and Jane; Joe likes Jane and Joan; Jean and Joan like each other. Draw a digraph illustrating these relationships between John, Joan, Jean, Jane and Joe.

1.7 Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders and insects. Draw a digraph representing this predatory behaviour.